

Extensions, interpolation and matching in \mathbb{R}^D

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Abstract

Suppose we are given an almost isometry $\phi : E \rightarrow \mathbb{R}^D$ where E is a finite subset of \mathbb{R}^D . Is it possible to decide if ϕ can be extended to an almost isometry to the whole of \mathbb{R}^D ? In this paper, we study this interesting question. It turns out that this question is also basic to problems of interpolation and matching in theoretical computer science. We study these connections as well.

Keywords and Phrases: Extension, Isometry, Interpolation, Data, Matching.

1 Introduction

Suppose we are given $\phi : E \rightarrow \mathbb{R}^D$ where $E \subset \mathbb{R}^D$ is a subset of \mathbb{R}^D . Is it possible to decide when ϕ extends smoothly to the whole of \mathbb{R}^D ? See [6, 8, 9, 10, 11, 3, 18, 20, 21, 22, 23, 24, 25, 26, 27, 28, 32, 34] and the references cited therein for a comprehensive account of this subject. Suppose now, we are given an almost isometry $\phi : E \rightarrow \mathbb{R}^D$ where E is a finite subset of \mathbb{R}^D . Can ϕ be extended to an almost isometry to the whole of \mathbb{R}^D ? In this paper, we study this interesting question. It turns out that this question is also basic to problems of interpolation and matching in theoretical computer science. We study these problems as well.

Let us begin by fixing some notations. In this paper, we will work in \mathbb{R}^D where $D \geq 2$ is fixed. Henceforth, by a Euclidean motion on \mathbb{R}^D , we shall mean a map $x \rightarrow Tx + x_0$ from \mathbb{R}^D to \mathbb{R}^D with $T \in O(D)$ or $T \in SO(D)$ and $x_0 \in \mathbb{R}^D$ a fixed column vector. $O(D)$ will be the orthogonal group of dimension D of isometries of \mathbb{R}^D that preserve a fixed point and $SO(D)$ the subgroup of $O(D)$ consisting of orientation preserving isometries of \mathbb{R}^D which preserve a fixed point. Equivalently, $O(D)$ is the group of orthogonal matrices of size $D \times D$ with determinant ± 1 generated by rotations and reflections and $SO(D)$ is the subgroup of orthogonal matrices of size $D \times D$ with determinant 1 generated by rotations. If $T \in SO(D)$, then the motion is proper. A Euclidean motion or more generally an invertible affine map is either proper or improper. Throughout, $|\cdot|$ denotes the usual Euclidean norm in \mathbb{R}^D .

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By a $k \geq 1$ point configuration in \mathbb{R}^D , we will henceforth mean a collection of points y_1, y_2, \dots, y_k in \mathbb{R}^D which are distinct. When we are given two k point configurations y_1, y_2, \dots, y_k and z_1, z_2, \dots, z_k in \mathbb{R}^D , we will assume that the y_i and z_i are distinct, $1 \leq i \leq k$.

An important problem in computer vision is recognizing points in \mathbb{R}^D . One way to think of this is as follows: Given two sets of points in \mathbb{R}^D do there exist combinations of rotations, translations, reflections and compositions of these which map the one set of points onto the other.

* A typical application of this problem arises in image processing where it is often necessary to align an unidentified image to an image from a given data base of images for example parts of the human body in face or fingerprint recognition. [†] See [12, 13, 14, 37, 36, 43, 2, 5, 41, 42, 48, 39] and the references cited therein for a broad perspective on applications of this problem. Thus the idea is to recognize points (often called landmarks) by verifying whether they align to points in the given data base. An image in \mathbb{R}^D does not change under Euclidean motions. In the case of labelled data (where the data points in each set are indexed by the same index set (for example k point configurations)), an old approach called the Procrustes approach [30, 31] analytically determines a Euclidean motion which maps the first configuration close to the other (in a L^2) sense. A statistical analysis of such methods can be found for example in [31]. There are a variety of ways to label points. See [12, 13, 14, 36, 43] and the references cited therein for a perspective. In this paper, we think of shape preservation in terms of whether there exists a Euclidean motion which maps one k point configuration onto a second. [‡]

One way to move past the Procrustes approach is to compare pairwise distances between labelled points. See for example [30, 31]. In this regard, the following result is well known. See for example [1, 44, 12].

Theorem 1.1 *Let y_1, \dots, y_k and z_1, \dots, z_k be two k point configurations in \mathbb{R}^D . Suppose that*

$$|z_i - z_j| = |y_i - y_j|, \quad 1 \leq i, j \leq k, \quad i \neq j.$$

Then there exists a Euclidean motion $\Phi : \mathbb{R}^D \rightarrow \mathbb{R}^D$ such that $\Phi(y_i) = z_i$, $i = 1, \dots, k$. If $k \leq D$, then Φ can be taken as proper.

Remark 1.2 *Other applications of matching point configurations from their pairwise distances are encountered for example in X-ray crystallography and in the mapping of restriction sites of DNA. See [37, 40] and the references cited therein. (In the case of one dimension, this problem is known as the turnpike problem or in molecular biology, it is known as the partial digest problem). See the work of [38, 36] which deals with algorithms and their running time for such matchings. We*

*In computer vision, two sets of points are said to have the same “shape” if there exist combinations of rotations, translations, reflections and compositions of these which map the one set of points onto the other.

[†]In computer vision an image is identified as a d dimensional point by stringing together the numbers associated to the pixels which construct it.

[‡]The shape identification problem can be stated more generally for other transformation groups. We restrict ourselves in this paper to Euclidean motions.

mention that a difficulty in trying to match point configurations is the absence of labels in the sense that often one does not know which point to map to which. We will not deal with the unlabeled problem in this paper.

Our first objective of this paper is to study alignment in the case where our pairwise distances are distorted and to do this we develop an analogy of Theorem 1.1 namely Theorem 3.4 and Theorem 3.5. This theory is motivated for example by noise considerations.

Our second objective in this paper concerns the connections between the alignment problem and the problem of extensions of almost isometries. More to the point, we are interested in studying the following question. Let $E \subset \mathbb{R}^D$ be a finite set. Given a function $\phi : E \rightarrow \mathbb{R}^D$ which is an almost isometry, we study the question of how to decide when ϕ extends to a smooth distorted isometry $\Phi : \mathbb{R}^D \rightarrow \mathbb{R}^D$ and we study how well Φ can be approximated by Euclidean motions on \mathbb{R}^D . This question depends on relations between the distortions of ϕ and Φ , the cardinality of E and the dimension D . Note that this question also is basic to interpolation and alignment.

The following result is known. See for example [44].

Theorem 1.3 *Let $E \subset \mathbb{R}^D$ and $\phi : E \rightarrow \mathbb{R}^D$ an isometry. Then ϕ extends to an isometry of \mathbb{R}^D onto \mathbb{R}^D .*

We shall prove Theorem 4.6 which may be viewed as an extension of Theorem 1.3. Several authors have studied extensions of isometries and almost isometries in \mathbb{R}^D . See [35, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 32] and the references cited therein.

Let $n \geq 1$. Suppose we are given $E \subset \mathbb{R}^n$, $\phi : E \rightarrow \mathbb{R}$ and an integer $m \geq 1$, how can one decide whether ϕ extends to a function $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}$ which is in $C^m(\mathbb{R}^n)$. Here $C^m(\mathbb{R}^n)$ denotes the space of functions $f : \mathbb{R}^n \rightarrow \mathbb{R}$ whose derivatives of order m are continuous and bounded. For $n = 1$ and E closed, this is the classical Whitney extension problem. More precisely in [45, 46, 47], H. Whitney proved the following: With the notation as above, ϕ extends to $C^m(\mathbb{R})$ if and only if the limiting values of all m th divided differences $[w_0, w_1, \dots, w_m]\phi$, where the $w_i \in E$ and $w_i \neq w_j$ if $i \neq j$, define a continuous function on the diagonal $\{w_0 = w_1 = \dots = w_m\}$. If $n = 1$, E is closed and ϕ is continuous, the Tietze's theorem provides the required extension to $C(\mathbb{R})$.

Progress on this problem was made by G. Glaeser [29], Y. Brudnyi and P. Shvartsman [6, 8, 9, 10, 11] and E. Bierstone, P. Milman and W. Pawlucki [3]. See also the work of N. Zobin [49, 50] for a related problem. Building on this work, C. Fefferman and his collaborators [18, 20, 21, 22, 23, 24, 25, 26, 27, 28, 32, 34] have given a complete solution to this problem and also to the problem with $C^m(\mathbb{R}^n)$ replaced by $C^{m,\omega}(\mathbb{R}^n)$, the space of functions whose m th derivatives have a given modulus of continuity ω as well as to the problem of C^m extension by linear operators. When E is taken to be finite, then these questions are basic to interpolation of data.

Throughout this paper, C, c, c_1, \dots denote positive constants depending only on D . These symbols need not denote the same constant in different occurrences. Throughout, we will use other

symbols to denote positive constants which may not depend on D or depend on D but also on other quantities. These symbols need not denote the same constant in different occurrences. Whenever, we suppose we are given a positive number ε , we assume ε is less than equal to a small enough positive constant determined by D .

Let $0 < \varepsilon < C$ and $\Phi : \mathbb{R}^D \rightarrow \mathbb{R}^D$ be a diffeomorphism. (In particular, Φ is one to one and onto). We say that Φ is “ ε -distorted” provided

$$(1 + \varepsilon)^{-1}I \leq [(\nabla\Phi)(x)]^T[(\nabla\Phi)(x)] \leq (1 + \varepsilon)I$$

as matrices, for all $x \in \mathbb{R}^D$. Here, I denotes the identity matrix in \mathbb{R}^D .

We will use the following properties of ε -distorted maps:

- If Φ is ε -distorted and $\varepsilon < \varepsilon'$, then Φ is ε' distorted.
- If Φ is ε -distorted, then so is Φ^{-1}
- If Φ and Ψ are ε -distorted, then $\Phi \circ \Psi$ is $C\varepsilon$ -distorted.
- Suppose Φ is ε -distorted. If τ is a piecewise smooth curve in \mathbb{R}^D , then the length of $\Phi(\tau)$ differs from that of τ by at most a factor of $(1 + \varepsilon)$. Consequently, if $x, x' \in \mathbb{R}^D$, then $|x - x'|$ and $|\Phi(x) - \Phi(x')|$ differ by at most a factor $(1 + \varepsilon)$.

2 Examples

In this section, we provide a few useful examples of ε -distorted diffeomorphisms of \mathbb{R}^D .

2.1 Slow twists

Example 1 Let $x \in \mathbb{R}^D$. Let $S(x)$ be the $D \times D$ identity matrix I_D or the $D \times D$ block-diagonal matrix

$$\begin{pmatrix} H_2(x) & 0 & 0 & 0 & 0 & 0 \\ 0 & H_3(x) & 0 & 0 & 0 & 0 \\ 0 & 0 & . & 0 & 0 & 0 \\ 0 & 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & 0 & . & 0 \\ 0 & 0 & 0 & 0 & 0 & H_D(x) \end{pmatrix}$$

where for $2 \leq i \leq D$ and a function f_i of one variable,

$$H_i(x) = \begin{pmatrix} \cos f_i(|x|) & \sin f_i(|x|) \\ -\sin f_i(|x|) & \cos f_i(|x|) \end{pmatrix}.$$

For example suppose $f_i \equiv 1$ for all i . If $D = 2$, then $S(x)$ is either I_2 or the rotation matrix

$$\begin{pmatrix} \cos(|x|) & \sin(|x|) \\ -\sin(|x|) & \cos(|x|) \end{pmatrix}.$$

Similarly, if $D = 3$, then $S(x)$ is either I_3 or the rotation matrix

$$\begin{pmatrix} \cos(|x|) & \sin(|x|) & 0 \\ -\sin(|x|) & \cos(|x|) & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is also easy to check the following: Suppose $f_i \equiv 1$ for all i . Then any fixed element of $O(D)$ with real entries is precisely $S(y)$ for some $y \in [0, 2\pi]$.

Define for each $x \in \mathbb{R}^D$, the map $\Phi(x) = (\Theta^T S(\Theta x))x$ where Θ is any fixed matrix in $SO(D)$. Let $0 < \varepsilon < C$. Then one checks that Φ is ε -distorted, provided for each i , $t|f'_i(t)| < c\varepsilon$ for all $t \in [0, \infty)$ and provided c is small enough. We call the map Φ a slow twist.

Example 2 Let $0 < \varepsilon < C$ and let $g : \mathbb{R}^D \rightarrow \mathbb{R}^D$ be a smooth map such that $|(\nabla g)(x)| < c\varepsilon$ for all $x \in \mathbb{R}^D$ and some $c > 0$. One checks that for each $x \in \mathbb{R}^D$, the map $\Phi(x) = x + g(x)$ is ε -distorted if c is small enough. We call the map Φ a slide (in analogy to translations).

The remainder of this paper is devoted to stating and proving our main results Theorem 3.4, Theorem 3.5 and Theorem 4.6.

We will make use of Whitney machinery developed by Fefferman in [18, 19, 20, 21, 22].

3 An extension of Theorem 1.1

In this section, we establish an extension of Theorem 1.1 for k point configurations whose pairwise Euclidean distances are distorted.

The main result of this section is the following theorem.

Theorem 3.4 *Given $0 < \varepsilon < C$ and $k \geq 1$, there exists $\delta > 0$ depending on ε small enough, such that the following holds. Let y_1, \dots, y_k and z_1, \dots, z_k be two k point configurations in \mathbb{R}^D . Suppose*

$$(1 + \delta)^{-1} \leq \frac{|z_i - z_j|}{|y_i - y_j|} \leq 1 + \delta, \quad i \neq j. \quad (3.1)$$

Then, there exists a Euclidean motion $\Phi_0 : x \rightarrow Tx + x_0$ such that

$$|z_i - \Phi_0(y_i)| \leq \varepsilon \text{diam} \{y_1, \dots, y_k\} \quad (3.2)$$

for each i . If $k \leq D$, then we can take Φ_0 to be a proper Euclidean motion on \mathbb{R}^D .

We can be more specific about the relationship between ε and δ in Theorem 3.4. This is contained in the following theorem.

Theorem 3.5 *Let $k \geq 1$. There exist constants $c_1, c_2 > 0$ depending on k and D such that the following holds. Let $0 < \varepsilon < C$. Let y_1, \dots, y_k and z_1, \dots, z_k be two k point configurations in \mathbb{R}^D scaled so that*

$$\sum_{i \neq j} |y_i - y_j|^2 + \sum_{i \neq j} |z_i - z_j|^2 = 1, \quad y_1 = z_1 = 0. \quad (3.3)$$

Suppose

$$||z_i - z_j| - |y_i - y_j|| < \varepsilon. \quad (3.4)$$

Then, there exists a Euclidean motion $\Phi_0 : x \rightarrow Tx + x_0$ such that

$$|z_i - \Phi_0(y_i)| \leq c_1 \varepsilon^{c_2} \quad (3.5)$$

for each i . If $k \leq D$, then we can take Φ_0 to be a proper Euclidean motion on \mathbb{R}^D .

We begin with the proof of Theorem 3.4.

Proof Suppose not. Then for each $l \geq 1$, we can find points $y_1^{(l)}, \dots, y_k^{(l)}$ and $z_1^{(l)}, \dots, z_k^{(l)}$ in \mathbb{R}^D satisfying (3.1) with $\delta = 1/l$ but not satisfying (3.2). Without loss of generality, we may suppose that $\text{diam} \{y_1^{(l)}, \dots, y_k^{(l)}\} = 1$ for each l and that $y_1^{(l)} = 0$ and $z_l^{(1)} = 0$ for each l . Thus $|y_i^{(l)}| \leq 1$ for all i and l and

$$(1 + 1/l)^{-1} \leq \frac{|z_i^{(l)} - z_j^{(l)}|}{|y_i^{(l)} - y_j^{(l)}|} \leq (1 + 1/l)$$

for $i \neq j$ and any l . However, for each l , there does not exist an Euclidean motion Φ_0 such that

$$|z_i^{(l)} - \Phi_0(y_i^{(l)})| \leq \varepsilon \quad (3.6)$$

for each i . Passing to a subsequence, l_1, l_2, l_3, \dots , we may assume

$$y_i^{(l_\mu)} \rightarrow y_i^\infty, \quad \mu \rightarrow \infty$$

and

$$z_i^{(l_\mu)} \rightarrow z_i^\infty, \quad \mu \rightarrow \infty.$$

Here, the points y_i^∞ and z_i^∞ satisfy

$$|z_i^\infty - z_j^\infty| = |y_i^\infty - y_j^\infty|$$

for $i \neq j$. Hence, by Theorem 1.1, there is an Euclidean motion Φ_0 such that $\Phi_0(y_i^\infty) = z_i^\infty$. Consequently, for μ large enough, (3.6) holds with l_μ . This contradicts the fact that for each l , there does not exist a Φ_0 satisfying (3.6) with l . Thus, we have proved all the assertions of the

theorem except that we can take Φ_0 to be proper if $k \leq D$. To see this, suppose that $k \leq D$ and let Φ_0 be an improper Euclidean motion such that

$$|z_i - \Phi_0(y_i)| \leq \varepsilon \text{diam} \{y_1, \dots, y_k\}$$

for each i . Then, there exists an improper Euclidean motion Ψ_0 that fixes y_1, \dots, y_k and in place of Φ_0 , we may use $\Psi_0 \circ \Phi_0$ in the conclusion of the Theorem. The proof of the Theorem is complete. \square .

We now prove Theorem 3.5.

Proof The idea of the proof relies on the following Łojasiewicz inequality, see [33]. Let $f : U \rightarrow \mathbb{R}$ be a real analytic function on an open set U in \mathbb{R}^D and Z be the zero locus of f . Assume that Z is not empty. Then for a compact set K in U , there exist positive constants α and α' depending on f and K such that for all $x \in K$, $|x - Z|^\alpha \leq \alpha' |f(x)|$.

Define

$$M := \{(X_1, \dots, X_k, Y_1, \dots, Y_k) : X_i, Y_i \in \mathbb{R}^D, ||X_i - X_j| - |Y_i - Y_j|| < \varepsilon\} \subset \mathbb{R}^{2kD}.$$

Define a map $F : \mathbb{R}^{2kD} \rightarrow \mathbb{R}$ as follows:

$$F : (X_1, X_2, \dots, X_k, Y_1, \dots, Y_k) \rightarrow \sum_{i \neq j} (|X_i - X_j|^2 - |Y_i - Y_j|^2)^2 : \mathbb{R}^{2kD} \rightarrow \mathbb{R}.$$

Then F is a polynomial and hence is real analytic on M . We claim that there exists $c_1 > 0$ depending on k, D such

$$F(y_1, \dots, y_k, z_1, \dots, z_k) \leq c_1 \varepsilon^2.$$

Indeed,

$$\begin{aligned} F(y_1, \dots, y_k, z_1, \dots, z_k) &\leq \sum_{i \neq j} (|y_i - y_j| - |z_i - z_j|)^2 (|y_i - y_j| + |z_i - z_j|)^2 \\ &\leq c_1 \varepsilon^2 \end{aligned}$$

so we have our claim. Let us now calculate that the zero set of F , $Z(F)$. Indeed, it is readily seen that

$$\begin{aligned} Z(F) &= \{x \in M : F(x) = 0\} \\ &= \{(X_1, \dots, X_k, Y_1, \dots, Y_k) \in M : |X_i - X_j| = |Y_i - Y_j|, \forall i \neq j\}. \end{aligned}$$

Using Theorem 1.1, there exists a Euclidean motion $\Phi_0 : \mathbb{R}^D \rightarrow \mathbb{R}^D$ such that $\Phi_0(p_i) = q_i$ for all i (and if $k \leq D$, Φ_0 can be taken to be proper).

We now apply Lojasiewicz's inequality to deduce the following: There exist $y'_1, \dots, y'_k, z'_1, \dots, z'_k \in \mathbb{R}^D$ and positive constants c_2, c_3 depending only on k, D such that

$$|(y_1, \dots, y_k, z_1, \dots, z_k) - (y'_1, \dots, y'_k, z'_1, \dots, z'_k)| \leq c_2 \varepsilon^{c_3}.$$

In particular, we have

$$|y_i - y'_i| \leq c_2 \varepsilon^{c_3}$$

and

$$|z_i - z'_i| \leq c_2 \varepsilon^{c_3}.$$

Now,

$$|\Phi_0(y_i) - \Phi_0(y'_i)| \leq c_4 \varepsilon^{c_3}$$

and $\Phi_0(y'_i) = z'_i$. So we have

$$|z_i - \Phi_0(y'_i)| \leq c_5 \varepsilon^{c_3}$$

and so consequently, we have

$$|\Phi_0(y_i) - z_i| \leq c_6 \varepsilon^{c_3}$$

which is what we needed to prove. \square

4 Smooth extensions of ε distorted maps

In this section we establish an extension of Theorem 1.3. This is the following theorem.

Theorem 4.6 *Let $0 < \varepsilon < C$ and $1 \leq k \leq D$. Then there exists $\delta > 0$ small enough depending on ε such that the following holds: Let y_1, \dots, y_k and z_1, \dots, z_k be two k point configurations in \mathbb{R}^D . Suppose that*

$$(1 + \delta)^{-1} \leq \frac{|z_i - z_j|}{|y_i - y_j|} \leq (1 + \delta), \quad 1 \leq i, j \leq k, \quad i \neq j.$$

Then there exists a diffeomorphism, 1-1 and onto map $\Phi : \mathbb{R}^D \rightarrow \mathbb{R}^D$ with

$$(1 + \varepsilon)^{-1} |x - y| \leq |\Phi(x) - \Phi(y)| \leq |x - y| (1 + \varepsilon), \quad x, y \in \mathbb{R}^D$$

satisfying

$$\Phi(y_i) = z_i, \quad 1 \leq i \leq k.$$

Remark 4.7 *In [15], we address the restriction $k \leq D$ in Theorem 4.6.*

We need two lemmas and a corollary which are consequences of the definitions of Slow twists and Slides.

As a consequence of Slow twists (see Example 1), we have the following:

Lemma 4.8 *Given $0 < \varepsilon < C$, there exists $\eta > 0$ small enough depending on ε for which the following holds. Let $\Theta \in SO(D)$ and let $0 < r_1 \leq \eta r_2$. Then, there exists an ε -distorted diffeomorphism $\Phi : \mathbb{R}^D \rightarrow \mathbb{R}^D$ such that*

$$\begin{cases} \Phi(x) = \Theta x, & |x| \leq r_1 \\ \Phi(x) = x, & |x| \geq r_2 \end{cases}$$

To see this, we illustrate the argument for $D = 2$ and $f_2 \equiv 1$ as the other cases are similar. We know that since $\Theta \in SO(2)$, $\Theta = H_1(y) = S(y)$ for some $y \in [0, 2\pi]$. Now we will define Φ as in Example 1. Clearly if $\Theta = I$, then $S = I$ and we are done. If not using orthogonality, it is easy to see that we require

$$H_1(\Theta x) = S(\Theta x) = \Theta^2 x$$

for $|x| \leq r_1$ and

$$H_1(\Theta x) = S(\Theta x) = \Theta x$$

for $|x| \geq r_2$. Thus the result follows.

As a consequence of Slides (see Example 2), we have the following:

Lemma 4.9 *Given $0 < \varepsilon < C$, there exists $\eta > 0$ small enough depending on ε such that the following holds. Let $\Phi_1 : x \rightarrow Tx + x_0$ be a proper Euclidean motion. Let $r_1, r_2 > 0$. Suppose $0 < r_1 \leq \eta r_2$ and $|x_0| \leq c\varepsilon r_1$. Then there exists an ε -distorted diffeomorphism $\Phi_2 : \mathbb{R}^D \rightarrow \mathbb{R}^D$ such that*

$$\begin{cases} \Phi_2(x) = \Phi_1(x), & |x| \leq r_1 \\ \Phi_2(x) = x, & |x| \geq r_2 \end{cases}$$

As a result of Lemma 4.8 and Lemma 4.9 we have the following corollary.

Corollary 4.10 *Given $0 < \varepsilon < C$, there exists $\eta > 0$ small enough depending on ε such that the following holds. Let $0 < r_1 \leq \eta r_2$ and let $x, x' \in \mathbb{R}^D$ with $|x - x'| \leq c\varepsilon r_1$ for some $c > 0$ and $|x| \leq r_1$. Then, there exists an ε -distorted diffeomorphism $\Phi : \mathbb{R}^D \rightarrow \mathbb{R}^D$ such that $\Phi(x) = x'$ and $\Phi(y) = y$ for $|y| \geq r_2$.*

We need to introduce a technique developed by Fefferman in [18, 19, 20, 21, 22] which involves the combinatorics of hierarchical clusterings of finite subsets of \mathbb{R}^D . In this regard, we need the following:

Lemma 4.11 *Let $k \geq 2$ be a positive integer and let $0 < \eta \leq 1/10$. Let $E \subset \mathbb{R}^D$ be a set consisting of k distinct points. Then, we can partition E into sets $E_1, E_2, \dots, E_{\mu_{\max}}$ and we can find a positive integer l ($10 \leq l \leq 100 + \binom{k}{2}$) such that the following hold:*

$$\text{diam}(E_\mu) \leq \eta^l \text{diam}(E) \tag{4.1}$$

for each μ and

$$\text{dist}(E_\mu, E_{\mu'}) \geq \eta^{l-1} \text{diam}(E), \text{ for } \mu \neq \mu'. \quad (4.2)$$

Proof We define an equivalence relation on E as follows. Define a relation \sim on E by saying that $x \sim x'$, for $x, x' \in E$ if and only if $|x - x'| \leq \eta^l \text{diam}(E)$ for a fixed positive integer l to be defined in a moment. By the pigeonhole principle, we can always find a positive integer l such that

$$|x - x'| \notin (\eta^l \text{diam}(E), \eta^{l-1} \text{diam}(E)], \quad x, x' \in E.$$

and such that $10 \leq l \leq 100 + \binom{k}{2}$. Let us choose and fix such an l and use it for \sim as defined above. Then \sim is an equivalence relation and the equivalence classes of \sim partition E into the sets $E_1, \dots, E_{\mu_{\max}}$ with the properties as required. \square

4.1 A special case of Theorem 4.6

In this section, we prove a special case of Theorem 4.6. This is given in the following theorem.

Theorem 4.12 *Let $0 < \varepsilon < C$, $k \geq 1$ and let m be a positive integer. Let $\lambda > 0$ be less than a small enough constant depending only on ε , m and D . Let $\delta > 0$ be less than a small enough constant depending only on λ , ε , m and D . Then the following holds: Let $E := y_1, \dots, y_k$ and $E' := z_1, \dots, z_k$ be distinct points in \mathbb{R}^D with $k \leq D$ and $y_1 = z_1$. Assume the following:*

$$|y_i - y_j| \geq \lambda^m \text{diam} \{y_1, \dots, y_k\}, \quad i \neq j \quad (4.3)$$

and

$$(1 + \delta)^{-1} \leq \frac{|z_i - z_j|}{|y_i - y_j|} \leq (1 + \delta), \quad i \neq j. \quad (4.4)$$

Then, there exists an ε -distorted diffeomorphism $\Phi : \mathbb{R}^D \rightarrow \mathbb{R}^D$ such that

$$\Phi(y_i) = z_i, \quad 1 \leq i \leq k \quad (4.5)$$

and

$$\Phi(x) = x \text{ for } |x - y_1| \geq \lambda^{-1/2} \text{diam} \{y_1, \dots, y_k\}. \quad (4.6)$$

Proof Without loss of generality, we may take $y_1 = z_1 = 0$ and $\text{diam} \{y_1, \dots, y_k\} = 1$. Applying Theorem 3.4 with $10^{-9} \varepsilon \lambda^{m+5}$ in place of ε , we obtain a proper Euclidean motion

$$\Phi_0 : x \rightarrow Tx + x_0 \quad (4.7)$$

such that

$$|\Phi_0(y_i) - z_i| \leq 10^{-9} \varepsilon \lambda^{m+5} \quad (4.8)$$

for each i . In particular, taking $i = 1$ and recalling that $y_1 = z_1 = 0$, we find that

$$|x_0| \leq 10^{-9} \varepsilon \lambda^{m+5}. \quad (4.9)$$

For each i , we consider the balls

$$B_i = B(\Phi_0(y_i), \lambda^{m+3}), B_i^+ = B(\Phi_0(y_i), \lambda^{m+1}). \quad (4.10)$$

Note that (4.3) shows that the balls B_i^+ have pairwise disjoint closures since Φ_0 is an Euclidean motion. Applying Corollary 4.10 to Lemma 4.9, we obtain for each i , a ε -distorted diffeomorphism $\Psi_i : \mathbb{R}^D \rightarrow \mathbb{R}^D$ such that

$$(\Psi_i \circ \Phi_0)(y_i) = z_i \quad (4.11)$$

and

$$\Psi_i(x) = x \quad (4.12)$$

outside B_i^+ . In particular, we see that

$$\Psi_i : B_i^+ \rightarrow B_i^+ \quad (4.13)$$

is one to one and onto. We may patch the Ψ_i together into a single map $\Psi : \mathbb{R}^D \rightarrow \mathbb{R}^D$ by setting

$$\Psi(x) := \begin{cases} \Psi_i(x), & x \in B_i^+ \\ x, & x \notin \cup_j B_j^+ \end{cases}. \quad (4.14)$$

Since the B_i^+ have pairwise disjoint closures (4.12) and (4.13) show that Ψ maps \mathbb{R}^D to \mathbb{R}^D and is one to one and onto. Moreover, since each Ψ_i is ε -distorted, it now follows easily that

$$\Psi : \mathbb{R}^D \rightarrow \mathbb{R}^D \quad (4.15)$$

is an ε -distorted diffeomorphism. From (4.10), (4.11) and (4.14), we also see also that

$$(\Psi \circ \Phi_0)(y_i) = z_i, \forall i. \quad (4.16)$$

Suppose $x \in \mathbb{R}^D$ with $|x| \geq 5$. Then (4.7) and (4.9) show that $|\Phi_0(x)| \geq 4$. On the other hand, each y_i satisfies

$$|y_i| = |y_i - y_1| \leq \text{diam} \{y_1, \dots, y_k\} = 1$$

so another application of (4.7) and (4.9) yields $|\Phi_0(y_i)| \leq 2$. Hence, $\Phi_0(x) \notin B_i^+$, see (4.10). Consequently, (4.14) yields

$$(\Psi \circ \Phi_0)(x) = \Phi_0(x), |x| \geq 5. \quad (4.17)$$

From (4.15), we obtain that

$$\Psi \circ \Phi : \mathbb{R}^D \rightarrow \mathbb{R}^D \quad (4.18)$$

is an ε -distorted diffeomorphism since Φ_0 is an Euclidean motion. Next, applying Lemma 4.9, with $r_1 = 10$ and $r_2 = \lambda^{-1/2}$, we obtain an ε -distorted diffeomorphism $\Psi : \mathbb{R}^D \rightarrow \mathbb{R}^D$ such that

$$\Psi_1(x) = \Phi_0(x), |x| \leq 10 \quad (4.19)$$

and

$$\Psi_1(x) = x, |x| \geq \lambda^{-1/2}. \quad (4.20)$$

Note that Lemma 4.9 applies, thanks to (4.9) and because we may assume $\frac{\lambda^{-1/2}}{10} > \eta^{-1}$ with η as in Lemma 4.9, thanks to our “small λ hypothesis”.

We now define

$$\tilde{\Psi}(x) := \begin{cases} (\Psi \circ \Phi_0)(x), & |x| \leq 10 \\ \Psi_1(x), & |x| \geq 5. \end{cases} \quad (4.21)$$

In the overlap region $5 \leq |x| \leq 10$, (4.17) and (4.19) show that $(\Psi \circ \Phi_0)(x) = \Phi_0(x) = \Psi_1(x)$ so (4.21) makes sense.

We now check that $\tilde{\Psi} : \mathbb{R}^D \rightarrow \mathbb{R}^D$ is one to one and onto. To do so, we introduce the sphere $S := \{x : |x| = 7\} \subset \mathbb{R}^D$ and partition \mathbb{R}^D into S , $\text{inside}(S)$ and $\text{outside}(S)$. Since $\Psi_1 : \mathbb{R}^D \rightarrow \mathbb{R}^D$ is one to one and onto, (4.19) shows that the map

$$\Psi_1 : \text{outside}(S) \rightarrow \text{outside}(\Phi_0(S)) \quad (4.22)$$

is one to one and onto. Also, since $\Psi \circ \Phi_0 : \mathbb{R}^D \rightarrow \mathbb{R}^D$ is one to one and onto, (4.17) shows that the map

$$\Psi \circ \Phi_0 : \text{inside}(S) \rightarrow \text{inside}(\Phi_0(S)) \quad (4.23)$$

is one to one and onto. In addition, (4.17) shows that the map

$$\Psi \circ \Phi_0 : (S) \rightarrow (\Phi_0(S)) \quad (4.24)$$

is one to one and onto. Comparing (4.21) with (4.22), (4.23) and (4.24), we see that $\tilde{\Psi} : \mathbb{R}^D \rightarrow \mathbb{R}^D$ is one to one and onto. Now since, also $\Psi \circ \Phi_0$ and Ψ_1 are ε -distorted, it follows at once from (4.21) that $\tilde{\Psi}$ is smooth and

$$(1 + \varepsilon)^{-1}I \leq (\nabla \tilde{\Psi}(x))^T \nabla \tilde{\Psi}(x) \leq (1 + \varepsilon)I, x \in \mathbb{R}^D.$$

Thus,

$$\tilde{\Psi} : \mathbb{R}^D \rightarrow \mathbb{R}^D \quad (4.25)$$

is an ε -distorted diffeomorphism. From (4.20), (4.21), we see that $\tilde{\Psi}(x) = x$ for $|x| \geq \lambda^{-1/2}$. From (4.16), (4.21), we have $\tilde{\Psi}(y_i) = z_i$ for each i , since, as we recall,

$$|y_i| = |y_i - y_1| \leq \text{diam} \{y_1, \dots, y_k\} = 1.$$

Thus, $\tilde{\Psi}$ satisfies all the assertions in the statement of the Theorem and the proof of the Theorem is complete. \square

4.2 Proof of Theorem 4.6

In this subsection, we prove Theorem 4.6. Theorem 4.6 follows immediately from the following theorem which is of independent interest.

Theorem 4.13 *Given $0 < \varepsilon < C$, there exist $\eta, \delta > 0$ small enough depending on ε such that the following holds. Let $E := \{y_1, \dots, y_k\}$ and $E' := \{z_1, \dots, z_k\}$ be points of \mathbb{R}^D with $1 \leq k \leq D$ and $y_1 = z_1$. Suppose*

$$(1 + \delta)^{-1} \leq \frac{|z_i - z_j|}{|y_i - y_j|} \leq (1 + \delta), \quad i \neq j. \quad (4.26)$$

Then, there exists an ε -distorted diffeomorphism $\Phi : \mathbb{R}^D \rightarrow \mathbb{R}^D$ such that

$$\Phi(y_i) = z_i \quad (4.27)$$

for each i and

$$\Phi(x) = x \quad (4.28)$$

for

$$|x - y_1| \geq \eta^{-1} \text{diam} \{y_1, \dots, y_k\}.$$

Proof We use induction on k . The case $k = 1$ is trivial, we can just take Φ to be the identity map. For the induction step, we fix $k \geq 2$ and suppose we already know the Theorem when k is replaced by $k' < k$. We will prove the Theorem for the given k . Let $\varepsilon > 0$ be given. We pick small positive numbers δ', λ, δ as follows.

$$\delta' < \alpha(\varepsilon, D). \quad (4.29)$$

$$\lambda < \alpha'(\delta', \varepsilon, D). \quad (4.30)$$

$$\delta < \alpha''(\lambda, \delta', \varepsilon, D). \quad (4.31)$$

Now let $y_1, \dots, y_k, z_1, \dots, z_k \in \mathbb{R}^D$ satisfy (4.26). We must produce an ε -distorted diffeomorphism $\Phi : \mathbb{R}^D \rightarrow \mathbb{R}^D$ satisfying (4.27) and (4.28) for some η depending only on $\delta, \lambda, \delta', \varepsilon, D$. That will complete the proof of the Theorem.

We apply Lemma 4.11 to $E = \{y_1, \dots, y_k\}$ with λ in place of η . Thus, we obtain an integer l and a partition of E into subsets $E_1, E_2, \dots, E_{\mu_{\max}}$ with the following properties:

$$10 \leq l \leq 100 + \binom{k}{2}. \quad (4.32)$$

$$\text{diam}(E_\mu) \leq \lambda^l \text{diam}(E) \quad (4.33)$$

for each μ .

$$\text{dist}(E_\mu, E_{\mu'}) \geq \lambda^{l-1} \text{diam}(E) \quad (4.34)$$

for $\mu \neq \mu'$. Note that

$$\text{card}(E_\mu) < \text{card}(E) = k \quad (4.35)$$

for each μ thanks to (4.33). For each μ , let

$$I_\mu := \{i : y_i \in E_\mu\}. \quad (4.36)$$

For each μ , we pick a “representative” $i_\mu \in I_\mu$. The $I_1, \dots, I_{\mu_{\max}}$ form a partition of $\{1, \dots, k\}$. Without loss of generality, we may suppose

$$i_1 = 1. \quad (4.37)$$

Define

$$I_{\text{rep}} := \{i_\mu : \mu = 1, \dots, \mu_{\max}\} \quad (4.38)$$

$$E_{\text{rep}} := \{y_{i_\mu} : \mu = 1, \dots, \mu_{\max}\}. \quad (4.39)$$

From (4.33), (4.34), we obtain

$$(1 - 2\lambda^l)\text{diam}(E) \leq \text{diam}(E_{\text{rep}}) \leq \text{diam}(E),$$

and

$$|x' - x''| \geq \lambda^{l-1}\text{diam}(E)$$

for $x, x' \in S_{\text{rep}}$, $x' \neq x''$. Hence,

$$(1/2)\text{diam}(E) \leq \text{diam}(E_{\text{rep}}) \leq \text{diam}(E) \quad (4.40)$$

and

$$|x' - x''| \geq \lambda^m \text{diam}(E_{\text{rep}}) \quad (4.41)$$

for $x', x'' \in E_{\text{rep}}$, $x' \neq x''$ where

$$m = 100 + \binom{D}{2}. \quad (4.42)$$

See (4.32) and recall that $k \leq D$. We now apply Theorem 4.12 to the points y_i , $i \in I_{\text{rep}}$, z_i , $i \in I_{\text{rep}}$ with ε in Theorem 4.12 replaced by our present δ' . The hypothesis of Theorem 4.12 holds, thanks to the smallness assumptions (4.30) and (4.31). See also (4.42), together with our present hypothesis (4.26). Note also that $1 \in I_{\text{rep}}$ and $y_1 = z_1$. Thus we obtain a δ' -distorted diffeomorphism $\Phi_0 : \mathbb{R}^D \rightarrow \mathbb{R}^D$ such that

$$\Phi_0(y_i) = z_i, \quad i \in I_{\text{rep}} \quad (4.43)$$

and

$$\Phi_0(x) = x \text{ for } |x - y_1| \geq \lambda^{-1/2}\text{diam}\{y_1, \dots, y_k\}. \quad (4.44)$$

Define

$$y'_i = \Phi_0(y_i), \quad i = 1, \dots, k. \quad (4.45)$$

Thus,

$$y'_{i_\mu} = z_{i_\mu} \quad (4.46)$$

for each μ and

$$(1 + C\delta')^{-1} \leq \frac{|z_i - z_j|}{|y'_i - y'_j|} \leq (1 + C\delta'), \quad i \neq j \quad (4.47)$$

thanks to (4.26), (4.31), (4.45) and the fact that $\Phi_0 : \mathbb{R}^D \rightarrow \mathbb{R}^D$ is a δ' -distorted diffeomorphism. Now fix μ ($1 \leq \mu \leq \mu_{\max}$). We now apply our inductive hypothesis with $k' < k$ to the points $y'_i, z_i, i \in I_\mu$. (Note that the inductive hypothesis applies, thanks to (4.35). Thus, there exists

$$\eta_{\text{indhyp}}(D, \varepsilon) > 0, \quad \delta_{\text{indhyp}}(D, \varepsilon) > 0 \quad (4.48)$$

such that the following holds: Suppose

$$(1 + \delta_{\text{indhyp}})^{-1} |y'_i - y'_j| \leq |z_i - z_j| \leq |y'_i - y'_j| (1 + \delta_{\text{indhyp}}), \quad i, j \in I_\mu \quad (4.49)$$

and

$$y'_{i_\mu} = z_{i_\mu}. \quad (4.50)$$

Then there exists a ε distorted diffeomorphism $\Psi_\mu : \mathbb{R}^D \rightarrow \mathbb{R}^D$ such that

$$\Psi_\mu(y'_i) = z_i, \quad i \in I_\mu \quad (4.51)$$

and

$$\Psi_\mu(x) = x, \quad \text{for } |x - y'_{i_\mu}| \geq \eta_{\text{indhyp}}^{-1} \text{diam}(S_\mu). \quad (4.52)$$

We may suppose $C\delta' < \delta_{\text{indhyp}}$ with C as in (4.47), thanks to (4.48) and our smallness assumption (4.29). Similarly, we may suppose that $\eta_{\text{indhyp}}^{-1} < 1/2\lambda^{-1/2}$, thanks to (4.48) and our smallness assumption (4.30). Thus (4.49) and (4.50) hold, by virtue of (4.47) and (4.46). Hence, for each μ , we obtain an ε -distorted diffeomorphism $\Psi_\mu : \mathbb{R}^D \rightarrow \mathbb{R}^D$, satisfying (4.51) and (4.52). In particular, (4.52) yields

$$\Psi_\mu(x) = x, \quad \text{for } |x - y'_{i_\mu}| \geq 1/2\lambda^{-1/2} \text{diam}(S_\mu). \quad (4.53)$$

Taking

$$B_\mu = B(y'_{i_\mu}, 1/2\lambda^{-1/2} \text{diam}(S_\mu)), \quad (4.54)$$

we see from (4.53), that

$$\Psi_\mu : B_\mu \rightarrow B_\mu \quad (4.55)$$

is one to one and onto since Ψ_μ is one to one and onto. Next, we note that the balls B_μ are pairwise disjoint. [§] This follows from (4.33), (4.34) and the definition (4.54). We may therefore define a map $\Psi : \mathbb{R}^D \rightarrow \mathbb{R}^D$ by setting

$$\Psi(x) := \begin{cases} \Psi_\mu(x), & x \in B_\mu, \text{ any } \mu \\ x, & x \notin \cup_\mu B_\mu \end{cases}. \quad (4.56)$$

Thanks to (4.55), we see that Ψ maps \mathbb{R}^D to \mathbb{R}^D one to one and onto. Moreover, since each Ψ_μ is an ε -distorted diffeomorphism satisfying (4.53), we see that Ψ is smooth on \mathbb{R}^D and that

$$(1 + \varepsilon)^{-1} I \leq (\nabla \Psi(x))^T (\nabla \Psi(x)) \leq (1 + \varepsilon) I, \quad x \in \mathbb{R}^D.$$

[§]Note that the closed ball B_μ is a single point if S_μ is a single point.

Thus,

$$\Psi : \mathbb{R}^D \rightarrow \mathbb{R}^D \quad (4.57)$$

is an ε -distorted diffeomorphism. From (4.51) and (4.56), we see that

$$\Psi(y'_i) = z_i, \quad i = 1, \dots, k. \quad (4.58)$$

Let us define

$$\Phi = \Psi \circ \Phi_0. \quad (4.59)$$

Thus

$$\Phi \text{ is a } C\varepsilon \text{ -distorted diffeomorphism of } \mathbb{R}^D \rightarrow \mathbb{R}^D \quad (4.60)$$

since $\Psi, \Phi_0 : \mathbb{R}^D \rightarrow \mathbb{R}^D$ are ε distorted diffeomorphisms. Also

$$\Phi(y_i) = z_i, \quad i = 1, \dots, k \quad (4.61)$$

as we see from (4.45) and (4.58). Now suppose that

$$|x - y_1| \geq \lambda^{-1} \text{diam} \{y_1, \dots, y_k\}.$$

Since $\Phi_0 : \mathbb{R}^D \rightarrow \mathbb{R}^D$ is a ε -distorted diffeomorphism, we have

$$|\Phi_0(x) - y'_1| \geq (1 + \varepsilon)^{-1} \lambda^{-1} \text{diam} \{y_1, \dots, y_k\} \quad (4.62)$$

and

$$\text{diam} \{y'_1, \dots, y'_k\} \leq (1 + \varepsilon) \text{diam} \{y_1, \dots, y_k\}.$$

See (4.45).

Hence for each μ ,

$$\begin{aligned} \left| \Phi_0(x) - y'_{i_\mu} \right| &\geq [(1 + \varepsilon)^{-1} \lambda^{-1} - (1 + \varepsilon)] \text{diam} \{y_1, \dots, y_k\} \\ &> 1/2 \lambda^{-1/2} \text{diam}(S_\mu). \end{aligned}$$

Thus, $\Phi_0(x) \notin \cup_\mu B_\mu$, see (4.54), and therefore $\Psi \circ \Phi_0 = \Phi_0$ as maps see (4.56). Thus,

$$\Phi(x) = \Phi_0(x). \quad (4.63)$$

From (4.44) and (4.63), we see that $\Phi(x) = x$. Thus, we have shown that

$$|x - y_1| \geq \lambda^{-1} \text{diam} \{y_1, \dots, y_k\}$$

implies $\Phi(x) = x$. That is, (4.28) holds with $\eta = \lambda$. Since also (4.60) and (4.61) hold we have carried out our inductive step completely and hence, the proof of the Theorem \square .

4.3 A Counterexample

In this final section, we show that in general one cannot expect to have Theorem 4.6 for $k > D$. This is contained in the following counterexample.

Counterexample: Fix $2D + 1$ points as follows. Let $\delta > 0$ be a small positive number depending on D . Let $y_1, \dots, y_{D+1} \in \mathbb{R}^D$ be the vertices of a regular simplex, all lying on the sphere of radius δ about the origin. Then define $y_{D+2} \dots y_{2D+1} \in \mathbb{R}^D$ such that y_{D+1}, \dots, y_{2D+1} are the vertices of a regular simplex, all lying in a sphere of radius 1, centered at some point $w_1 \in \mathbb{R}^D$. Next, we define a map

$$\phi : \{y_1, \dots, y_{2D+1}\} \rightarrow \{y_1, \dots, y_{2D+1}\}$$

as follows. We take $\phi|_{\{y_1, \dots, y_{D+1}\}}$ to be an odd permutation that fixes y_{D+1} , and take $\phi|_{\{y_{D+1}, \dots, y_{2D+1}\}}$ to be the identity. The map ϕ distorts distances by at most a factor $1 + C\delta$. Here, we can take δ arbitrarily small. On the other hand, for small enough $\varepsilon_0 > 0$ depending only on D , we will show that ϕ cannot be extended to a map $\phi : \mathbb{R}^D \rightarrow \mathbb{R}^D$ satisfying

$$(1 + \varepsilon_0)^{-1} |x - x'| \leq |\Phi(x) - \Phi(x')| \leq |x - x'| (1 + \varepsilon_0), \quad x, x' \in \mathbb{R}^D.$$

In fact, suppose that such a Φ exists. Then Φ is continuous. Note that there exists $T \in O(D)$ with $\det T = -1$ such that $\phi(y_i) = Ty_i$ for $i = 1, \dots, D + 1$. Let S_t be the sphere of radius $r_t := \delta \cdot (1 - t) + 1 \cdot t$ centered at $t \cdot w_1$ for $t \in [0, 1]$ and let S'_t be the sphere of radius r_t centered at $\Phi(t \cdot w_1)$. Also, let Sh_t be the spherical shell

$$\{x \in \mathbb{R}^D : r_t \cdot (1 + \varepsilon_0)^{-1} \leq |x - \Phi(t \cdot w_1)| \leq r_t \cdot (1 + \varepsilon_0)\}$$

and let $\pi_t : Sh_t \rightarrow S'_t$ be the projection defined by

$$\pi_t(x) - \Phi(t \cdot w_1) = \frac{x - \Phi(t \cdot w_1)}{|x - \Phi(t \cdot w_1)|} \cdot r_t.$$

Since Φ agrees with ϕ , we know that

$$|\Phi(x) - Tx| \leq C\varepsilon_0\delta, \quad |x| = \delta. \quad (4.64)$$

Since Φ agrees with ϕ , we know that

$$|\Phi(x) - x| \leq C\varepsilon_0, \quad |x - w_1| = 1. \quad (4.65)$$

Our assumption that Φ is an approximate isometry shows that

$$\Phi : S_t \rightarrow Sh_t, \quad 0 \leq t \leq 1$$

and

$$(\pi_t) \circ (\Phi) : S_t \rightarrow S'_t, \quad 0 \leq t \leq 1. \quad (4.66)$$

We can therefore define a one-parameter family of maps Ψ_t , $t \in [0, 1]$ from the unit sphere to itself by setting

$$\Psi_t(w) = \frac{(\pi_t \circ \Phi)(tw_1 + r_t w) - \Phi(tw_1)}{|(\pi_t \circ \Phi)(tw_1 + r_t w) - \Phi(tw_1)|} = \frac{(\pi_t \circ \Phi)(tw_1 + r_t w) - \Phi(tw_1)}{r_t}$$

from the unit sphere to itself. Then Ψ_t is a continuous family of continuous maps from the unit sphere to itself. From (4.64), we see that Ψ_0 is a small perturbation of the map $T : S^{D-1} \rightarrow S^{D-1}$ which has degree -1. From (4.65), we see that Ψ_1 is a small perturbation of the identity.

Consequently, the following must hold:

- Degree Ψ_t is independent of $t \in [0, 1]$.
- Degree $\Psi_0 = -1$.
- Degree $\Psi_1 = +1$.

Thus, we have arrived at a contradiction, proving that there does not exist an extension Φ as above. \square

Remark 4.14 *The counter example above could be motivated by the fact that the mapping swapping the numbers δ and $-\delta$ and fixing the number 1 cannot be extended to a continuous bijection of the line.*

Remark 4.15 *The case of $k > D$ absent in Theorem 4.6 is investigated by us in the paper [15] where it is shown that we may allow for this case if roughly we require that on any $D + 1$ of the k points which form a relatively voluminous simplex, the extension Φ is orientation preserving. Other conditions concerning when the extension does not exist are also studied and an explicit exponential dependence of δ and ε is given.*

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References

- [1] P. Alestalo, D.A. Trotsenko and J. Vaisala, *The linear extension property of bi-Lipchitz mappings*, Siberian Math. J, **44**(6)(2003), pp 959-968.
- [2] A.S. Bandeira, M. Charikar, A. Singer, A. Zhu, *Multireference Alignment using Semidefinite Programming*, 5th Innovations in Theoretical Computer Science (2014).
- [3] E. Bierstone, P. Milman, W. Pawblucki, *Differentiable functions defined on closed sets: A problem of Whitney*, Inventiones Math **151**(2)(2003), pp 329-352.
- [4] E. Bierstone, P. Milman, W. Pawblucki, *High-order tangents and Fefferman's paper on Whitney's extension problem*, Annals of Mathematics, **164**(2006), pp 361-370.
- [5] N. Boumal, A. Singer, P.A. Absil and V. D. Blondel, *Cramer-Rao bounds for synchronization of rotations*, Information and Inference: A Journal of the IMA, **3**(1)(2014), pp 1-39.
- [6] Y. Brudnyi, *On an extension theorem*, Funk. Anal. i Prilzhen, **4**(1970), pp 243-252.

- [7] Y. Brudnyi and P. Shvartsman, *The traces of differentiable functions to closed subsets on \mathbb{R}^n* , in Function Spaces (Poznan, 1989), Teubner-Texte Math, **120**, Teubner, Stuttgart (1991), pp 206-210.
- [8] Y. Brudnyi and P. Shvartsman, *A linear extension operator for a space of smooth functions defined on subsets of \mathbb{R}^n* , Dokl. Akad. Nauk SSSR **280**(1985), pp 268-270. English transl. in Soviet math. Dokl. **31**, No 1 (1985), pp 48-51.
- [9] Y. Brudnyi and P. Shvartsman, *Generalizations of Whitney's extension theorem*, Int. Math. Research Notices **3**(1994), pp 515-574.
- [10] Y. Brudnyi and P. Shvartsman, *The Whitney problem of existence of a linear extension operator*, J. Geometric Analysis, **7**(4)(1997), pp 515-574.
- [11] Y. Brudnyi and P. Shvartsman, *Whitneys extension problem for multivariate $C^{(1,w)}$ functions*, Trans. Amer. Math. Soc, **353**(6)(2001), pp 2487-2512.
- [12] M. Boutin and G. Kemper, *Which point configurations are determined by the distribution of their pairwise distances?*, Int. J. Compt. Geometry and Appl, **17**(1)(2007), pp 31-43.
- [13] M. Boutin and G. Kemper, *On reconstructing configurations of points in P^2 from a joint distribution of invariants*, Applicable Algebra in Engineering, Communication and Computing, **15**(6)(2005), pp 361-391.
- [14] M. Boutin and G. Kemper, *On reconstructing n -point configurations from the distribution of distances or areas*, Adv. Appl. Math. **32**(2004), pp 709-735.
- [15] S. B. Damelin and C. Fefferman *Extensions in \mathbb{R}^D* , preprint.
- [16] S. B. Damelin and C. Fefferman *On Extensions of ε Diffeomorphisms*, preprint.
- [17] C. Fefferman, S. B. Damelin and W. Glover, *BMO Theorems for ε distorted diffeomorphisms on \mathbb{R}^D and an application to comparing manifolds of speech and sound*, Involve **5-2**(2012), pp 159-172.
- [18] C. Fefferman, *Interpolation and extrapolation of smooth functions by linear operators*, Revista Matematica Iberoamericana, **21**(1)(2005), pp 313-348.
- [19] C. Fefferman, *A sharp form of Whitney's extension theorem*, Annals of Math, **161**(1)(2005), pp 509-577.
- [20] C. Fefferman, *Whitneys extension problem for C^m* , Annals of Math, **164**(1)(2006), pp 313-359.
- [21] C. Fefferman, *C^m extension by linear operators*, Annals of Math, **166**(3)(2007), pp 779-835.
- [22] C. Fefferman, *Whitney's extension problems and interpolation of data*, Bull. Amer. Math. Soc.(N.S), **46**(2)(2009), pp 207-220.
- [23] C. Fefferman and B. Klartag, *An example related to Whitney extension with almost minimal C^m norm*, Rev. Mat. Iberoamericana, **25**(2)(2009), pp 423-446.

- [24] C. Fefferman and B. Klartag, *Fitting a C^m Smooth Function to Data I*, Ann. of Math, **169**(1)(2009), pp 315-346.
- [25] C. Fefferman and B. Klartag, *Fitting a C^m Smooth Function to Data II*, Rev. Mat. Iberoamericana, **25**(1)(2009), pp 49-273.
- [26] C. Fefferman and B. Klartag, *An example related to Whitney extension with minimal C^m norm*, Rev. Mat. Iberoam, **25**(2)(2009), pp 423-446.
- [27] C. Fefferman and A. Israel, *The Jet of an Interpolant on a Finite Set*, Rev. Mat. Iberoamericana, **27**(1)(2011), pp 355-360.
- [28] C. Fefferman and A. Israel, *Sobolev Extension by Linear Operators*, J. Amer. Math. Soc, to appear.
- [29] G. Glaeser, *Etudes de quelques algebres tayloriennes*, J d'Analyse **6**(1958), pp 1-124.
- [30] J. Gower, *Generalized Procrustes analysis*, Psychometrika, **40**(1975), pp 33-51.
- [31] J. C. Gower, *Generalized Procrustes in the statistical analysis of shape*, J. Roy. Statist. Soc. Ser. B, **53**(2)(1991), pp 285-339.
- [32] A. Israel, *Bounded Linear Extension Operator for $L^2(\mathbb{R}^2)$* , Annals of Mathematics, **171**(1)(2013), pp 183-230.
- [33] Ji, Shanyu, Kollár, János and Shiffman, Bernard (1992), *A global Łojasiewicz inequality for algebraic varieties*, Transactions of the American Mathematical Society, **329**(2), pp 813-818.
- [34] B. Klartag and N. Zobin, *C^1 extensions of functions and stabilization of Glaeser refinements*, Revisita Math. Iberoamericana, **23**(2)(2007), pp 635-669.
- [35] E. LeGruyer, *Minimal Lipschitz extensions to differential functions defined on a Hilbert space*, Geometric and Functional Analysis, **19**(4)(2009), pp 1101-1118.
- [36] P. Lemke and M. Werman, *On the complexity of inverting the autocorrelation function of a finite integer sequence and the problem of locating n points on a line given the unlabeled distances between them*, Preprint **453**, IMA, 1988.
- [37] A. L. Patterson, *A direct method for the determination of the components of interatomic distances in crystals*, Zeitschr. Krist, **90**(1935), pp 517-542.
- [38] J. Rosenblatt and P. D. Seymour, *The structure of homometric sets*, SIAM J. Algebraic Discrete Methods **3**(1982), pp 343-350.
- [39] A. Singer and H.T. Wu, *Two-Dimensional tomography from noisy projections taken at unknown random directions*, SIAM Journal on Imaging Sciences, **6**(1)(2013), pp 136-175.
- [40] S.S. Skiena, W. D. Smith and P. Lemke, *Reconstructing sets from interpoint distances*, Discrete and computational geometry. The Goodman-Pollack Festschrift, Volume 25 of Algorithms and Combinatorics, Springer, Berlin, Berlin 2003, pp 597-632.

- [41] L. Wang, A. Singer, Z. Wen, *Orientation Determination from Cryo-EM images using Least Unsquared Deviations*, SIAM Journal on Imaging Sciences, **6**(4)(2013), pp 2450-2483.
- [42] L. Wang, A. Singer, *Exact and Stable Recovery of Rotations for Robust Synchronization*, Information and Inference: A Journal of the IMA, **2**(2)(2013), pp 145-193.
- [43] M. Werman and D. Weinshall, *Similarity and Affine Invariant Distance Between Point Sets*, PAMI, **17**(8), pp 810-814.
- [44] J. H. Wells and L. R. Williams, *Embeddings and extensions in analysis*, Ergebnisse der Mathematik und ihrer Grenzgebiete Band 84, Springer-Verlag, New York-Heidelberg, 1975.
- [45] H. Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. **36**(1934), pp 63-89.
- [46] H. Whitney, *Differentiable functions defined in closed sets I*, Trans. Amer. Math. Soc. **36**(1934), pp 369-389.
- [47] H. Whitney, *Functions differentiable on the boundaries of regions*, Annals of Math. **35**(1934), pp 482-485.
- [48] Z. Zhao and A. Singer, *Fourier-Bessel Rotational Invariant Eigenimages*, The Journal of the Optical Society of America A, **30**(5)(2013), pp 871-877.
- [49] N. Zobin, *Whitney's problem on extendability of functions and an intrinsic metric*, Advances in Math, **133**(1)(1998), pp 96-132.
- [50] N. Zobin, *Extension of smooth functions from finitely connected planar domains*, Journal of Geom. Analysis, **9**(3)(1999), pp 489-509.